

# A SHORT NOTE ON SIGN CHANGES

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**ABSTRACT.** In this paper, we present a quantitative result for the number of sign changes for the sequences  $\{a(n^j)\}_{n \geq 1}$ ,  $j = 2, 3, 4$  of the Fourier coefficients of normalized Hecke eigen cusp forms for the full modular group  $SL_2(\mathbb{Z})$ . We also prove a similar kind of quantitative result for the number of sign changes of the  $q$ -exponents  $c(p)$  ( $p$  vary over primes) of certain generalized modular functions for the congruence subgroup  $\Gamma_0(N)$ , where  $N$  is square-free.

## 1. INTRODUCTION

The sign changes of the Fourier coefficients of modular forms are well studied. It is known that, if the Fourier coefficients of a cusp form are real, then they change signs infinitely often. The proof of this fact follows from the Landau's theorem on Dirichlet series with non-negative coefficients. Further, many quantitative results for the number of sign changes for the sequence of the Fourier coefficients have been established. The sign changes of the subsequence of the Fourier coefficients at prime numbers was first studied by M. Ram Murty [13].

In our first result, we show that if  $h(z) = \sum_{n \geq 1} a(n)q^n$  is a Hecke eigenform for the full modular group  $SL_2(\mathbb{Z})$ , then the subsequences  $\{a(n^j)\}_{n \geq 1}$ ,  $j = 2, 3, 4$  of the Fourier coefficients change signs infinitely often. In fact, we give a lower bound for the number of sign changes in the interval  $(x, 2x]$ . We use the results of O. M. Fomenko [3], H. Lao and A. Sankaranarayanan [10] and G. S. Lü [11] to prove this result.

Our second result is about the sign changes of the  $q$ -exponents of a certain class of generalized modular functions  $f$  on the Hecke congruence subgroup  $\Gamma_0(N)$ . We always assume that  $f$  is non-constant and the weight of  $f$  is zero. A generalized modular function  $f$  is a holomorphic function on the upper half plane  $\mathcal{H}$ , meromorphic at the cusps, that transforms under  $\Gamma_0(N)$  like a usual modular function, with the exception that the character  $\chi$  need not be unitary, but  $\chi(\gamma) = 1$  for all parabolic  $\gamma \in \Gamma_0(N)$  of trace 2. For more details on generalized modular functions, see [7] and [8]. It is known that, every generalized modular function has a product

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expansion

$$f(z) = cq^h \prod (1 - q^n)^{c(n)} \quad (0 < |q| < \epsilon),$$

where  $h \in \mathbb{Z}$ ,  $c$  and  $c(n)$  are uniquely determined complex numbers and  $q = e^{2\pi iz}$  ( $z \in \mathcal{H}$ ). In [9], it has been proved that if  $f$  is a generalized modular function with  $\text{div}(f) = \emptyset$ , then  $c(n)$  ( $n \geq 1$ ) change signs infinitely often. This result was proved by using a theorem of [7] that any generalized modular function  $f$  with  $\text{div}(f) = \emptyset$  corresponds to a cusp form of weight 2 by taking the logarithmic derivative. However, the result is not quantitative. Using the results of Y. J. Choie and W. Kohnen [1], H. Iwaniec, W. Kohnen and J. Sengupta [6] and the relation between the  $q$ -exponents of generalized modular function and the Fourier coefficients of the corresponding cusp form of weight 2, one can deduce a bound for the first sign change of the  $q$ -exponents of a certain kind of generalized modular function.

In our second result, we give a quantitative result for the number of sign changes of the subsequence of  $q$ -exponents at primes,  $\{c(p)\}$ , in the interval  $(x, 2x]$  for a certain kind of generalized modular function on  $\Gamma_0(N)$ , where  $N$  is square-free. To prove this result, we use the method given by Y. J. Choie, A. Sankaranarayanan and J. Sengupta [2].

## 2. STATEMENT OF RESULTS

Let  $h(z) = \sum_{n \geq 1} a(n)q^n$  be a Hecke eigenform of weight  $k$  on the full modular group  $SL_2(\mathbb{Z})$ . Let  $\lambda(n) = \frac{a(n)}{n^{(k-1)/2}}$ . Denote by  $\delta_j = 2/11, 1/9, 2/27$  for  $j = 2, 3, 4$  respectively. Now, we state our first result.

**Theorem 2.1.** *For any  $j \in \{2, 3, 4\}$ , the sequence  $\{\lambda(n^j)\}_{n \geq 1}$  change signs infinitely often. Moreover, the sequence has at least  $\gg x^{\delta_j - 2\epsilon}$  sign changes in the interval  $(x, 2x]$  for sufficiently large  $x$ , where  $\epsilon$  is any small positive constant.*

From [7, Theorem 2], it is known that if  $f$  is a generalized modular function on  $\Gamma_0(N)$  with  $\text{div}(f) = \emptyset$ , then its logarithmic derivative  $g = \frac{1}{2\pi i} \frac{f'}{f}$  is a cusp form of weight 2 on  $\Gamma_0(N)$  with trivial character. We now state our second result.

**Theorem 2.2.** *Let  $N$  be a square-free positive integer and  $f$  be a non-constant generalized modular function on  $\Gamma_0(N)$  with  $\text{div}(f) = \emptyset$ . Suppose that the logarithmic derivative  $g = \frac{1}{2\pi i} \frac{f'}{f}$  is a normalized newform. Then the sequence  $c(p)$  ( $p$  prime) change signs infinitely often and it has at least  $\gg e^{A\sqrt{\log x}}$  sign changes in the interval  $(x, 2x]$  for sufficiently large  $x$ , where  $A$  is an absolute constant.*

## 3. PROOFS

**Proof of Theorem 2.1** Let us denote  $1/2, 3/4$  and  $7/9$  by  $\beta_j$  for  $j = 2, 3$  and  $4$  respectively. For any  $\epsilon > 0$ , [3] and [11] give the following estimates.

$$(1) \quad \sum_{n \leq x} \lambda(n^j) \ll_{f, \epsilon} x^{\beta_j + \epsilon}.$$

From [10, Theorem 1.1, 1.2, 1.3], we get the following estimates.

$$(2) \quad \sum_{n \leq x} \lambda^2(n^j) = B_j x + O_{f, \epsilon}(x^{1-\delta_j+\epsilon}),$$

where  $B_j$  are absolute constants,  $\delta_j$  are defined as before and these estimates are valid for any  $\epsilon > 0$ . Let  $h = h(x) = x^{1-\delta_j+2\epsilon}$ , where  $\epsilon$  is sufficiently small. We assume that the sequence  $\{\lambda(n^j)\}_{n \geq 1}$  are of constant sign say positive for all  $n \in (x, x+h]$ . Using (2), we get

$$(3) \quad \sum_{x < n \leq x+h} \lambda^2(n^j) = B_j h + O_{f, \epsilon}(x^{1-\delta_j+\epsilon}) \gg x^{1-\delta_j+2\epsilon}.$$

On the other hand, using (1), we get

$$(4) \quad \sum_{x < n \leq x+h} \lambda^2(n^j) = \sum_{x < n \leq x+h} \lambda(n^j) \lambda(n^j) \ll x^{2\epsilon} \sum_{x < n \leq x+h} \lambda(n^j) \ll x^{2\epsilon} ((x+h)^{\beta_j+\epsilon} + x^{\beta_j+\epsilon}) \\ \ll x^{\beta_j+3\epsilon}.$$

Now comparing  $1-\delta_j$  and  $\beta_j$  for  $j = 2, 3, 4$ , we see that the bounds in (3) and (4) for  $\sum_{x < n \leq x+h} \lambda^2(n^j)$  contradict each other. Therefore, at least one  $\lambda(n^j)$  for  $x < n \leq x+h$

must be negative. Hence the sparse sequences  $\{\lambda(n^j)\}_{n \geq 1}$  for  $j = 2, 3, 4$  change signs infinitely often and there are at least  $\gg x^{\delta_j-2\epsilon}$  sign changes in the interval  $(x, 2x]$ .

**Proof of Theorem 2.2** Since  $\text{div}(f) = \emptyset$ , we have  $f(z) = \prod_{n \geq 1} (1 - q^n)^{c(n)}$ . Let  $g(z) = \sum_{n \geq 1} b(n) q^n$  and  $\lambda(n) = \frac{b(n)}{n^{1/2}}$ . Then we have

$$b(n) = - \sum_{d|n} dc(d).$$

In particular,  $b(1) = 1 = -c(1)$  and  $b(p) = 1 - pc(p)$  for any prime  $p$ .

Using a theorem of C. J. Moreno ([12]) and a result of J. Hoffstein and D. Ramakrishnan ([5]) about the nonexistence of Siegel zero, we have the following estimate.

$$(5) \quad \sum_{p \leq x} \lambda(p) \log p = O(xe^{-A_1 \sqrt{\log x}}),$$

where  $A_1$  is an absolute constant. Following the proof of [14, Theorem 4.1] and using a nonvanishing result for symmetric square  $L$ -function of a newform on square-free

level due to D. Goldfeld, J. Hoffstein and D. Liemann ([4]) for the proof of [14, Lemma 4.1], we get the following estimate.

$$(6) \quad \sum_{p \leq x} \lambda^2(p) \log p = x + O(xe^{-A_2 \sqrt{\log x}}),$$

where  $A_2$  is an absolute constant.

Since  $\lambda(p) = \frac{b(p)}{\sqrt{p}} = -\left(\frac{-1}{\sqrt{p}} + \sqrt{p}c(p)\right)$ , therefore  $\sqrt{p}c(p) = \frac{1}{\sqrt{p}} - \lambda(p)$ . Let  $c'(p) = \sqrt{p}c(p)$  for all primes  $p$ . The behaviour of  $c'(p)$  and  $c(p)$  are same in the sense of sign changes since  $\frac{c'(p)}{c(p)}$  is a positive real number. We have

$$\begin{aligned} \sum_{p \leq x} c'(p) \log p &= \sum_{p \leq x} \left( \frac{1}{\sqrt{p}} - \lambda(p) \right) \log p \\ &= O(\sqrt{x} \log x) + O(xe^{-A_1 \sqrt{\log x}}), \end{aligned}$$

here we used (5) for getting the last line. Hence

$$(7) \quad \sum_{p \leq x} c'(p) \log p = O(xe^{-A_3 \sqrt{\log x}}), \quad A_3 \text{ is an absolute constant.}$$

We have

$$\begin{aligned} \sum_{p \leq x} c'^2(p) \log p &= \sum_{p \leq x} \left( \frac{1}{\sqrt{p}} - \lambda(p) \right)^2 \log p \\ &= \sum_{p \leq x} \lambda^2(p) \log p + 2 \sum_{p \leq x} \frac{c'(p) \log p}{p} - \sum_{p \leq x} \frac{\log p}{p}. \end{aligned}$$

Now, estimating first and second terms by using (6) and (7) respectively, we get the following.

$$(8) \quad \sum_{p \leq x} c'^2(p) \log p = x + O(xe^{-A_4 \sqrt{\log x}}), \quad A_4 \text{ is an absolute constant.}$$

Let  $h = h(x)$  be any function of  $x$  with the property that  $0 < h(x) < x$ . We evaluate  $\sum_{x < p \leq x+h} c'(p)$  by two different ways and arrive at a contradiction if  $c'(p)$  do

not change the sign as stated in the result. Assume that the  $c'(p)$  are of constant sign for  $x < p \leq x+h$ . Without loss of generality, we assume that  $c'(p) \geq 0$  for  $x < p \leq x+h$ . Using the fact that  $|c'(p)| \leq 3$ , we get the following.

$$(9) \quad \sum_{x < p \leq x+h} c'(p) \geq A_5 \sum_{x < p \leq x+h} c'^2(p),$$

where  $A_5$  is an absolute constant. Using the estimate (8), we have

$$\begin{aligned}
\sum_{p \leq x} c'^2(p) &= \sum_{p \leq x} c'^2(p) \frac{\log p}{\log p} \\
&= \frac{1}{\log x} \left( \sum_{p \leq x} c'^2(p) \log p \right) + \int_2^x \frac{\left( \sum_{p \leq t} c'^2(p) \log p \right)}{t \log^2 t} dt \\
&= \frac{1}{\log x} (x + O(xe^{-A_4 \sqrt{\log x}})) + \int_2^x \frac{t + O(te^{-A_4 \sqrt{\log t}})}{t \log^2 t} dt \\
&= \frac{x}{\log x} + O\left(\frac{x}{\log x} e^{-A_4 \sqrt{\log x}}\right) + \int_2^x \frac{1}{\log^2 t} dt + O\left(\int_2^x \frac{e^{-A_4 \sqrt{\log t}}}{\log^2 t} dt\right).
\end{aligned}$$

From the above, we get

$$(10) \quad \sum_{p \leq x} c'^2(p) \sim \frac{x}{\log x}.$$

Combining (9) and (10), we deduce

$$(11) \quad \sum_{x < p \leq x+h} c'(p) \geq A_5 \left( \sum_{p \leq x+h} c'^2(p) - \sum_{p \leq x} c'^2(p) \right) \geq A_6 \left( \frac{x+h}{\log(x+h)} - \frac{x}{\log x} \right) \gg \frac{h}{\log x},$$

where  $A_6$  is an absolute constant.

On the other hand using (7), we have the following estimate.

$$\begin{aligned}
\sum_{x < p \leq x+h} c'(p) &= \sum_{x < p \leq x+h} c'(p) \frac{\log p}{\log p} \\
&\leq \frac{1}{\log x} \sum_{x < p \leq x+h} c'(p) \log p \\
&\ll \frac{1}{\log x} \left( (x+h)e^{-A_3 \sqrt{\log(x+h)}} + xe^{-A_3 \sqrt{\log x}} \right).
\end{aligned}$$

From the above estimate, we obtain

$$(12) \quad \sum_{x < p \leq x+h} c'(p) \ll \frac{x}{\log x} e^{-A_3 \sqrt{\log x}}.$$

Choosing an appropriate absolute constant  $A$  and  $h(x) = \frac{x}{e^{A\sqrt{\log x}}}$ , we see that the bounds obtained in (11) and (12) contradict each other. Hence, at least one  $c'(p)$  for  $x < p \leq x+h$  must be negative. This implies that the sequence  $\{c(p)\}$  has infinitely many sign changes and there are at least  $\gg e^{A\sqrt{\log x}}$  sign changes for the

sequence  $\{c(p)\}$ , whenever  $p \in (x, 2x]$ . □

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